Faculty of Science, Technology, Engineering and Mathematics
M208 Pure mathematics

M208

TMA 01 2019J

Covers Book A **Cut-off date 7 November 2019**

You can submit your TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on the TMA, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' tab of the M208 website.

In the wording of the questions:

- *write down*, *list* or *state* means 'write down without justification' (unless otherwise stated)
- *find*, *determine*, *calculate*, *derive*, *evaluate* or *solve* means 'show all your working'
- *prove*, *show*, *deduce* or *verify* means 'justify each step'
- *sketch* means 'sketch without justification' and *describe* means 'describe without justification' (both unless otherwise stated).

In particular, if you use a definition, result or theorem to go from one line to the next, then you must state clearly which fact you are using – for example, you could quote the relevant unit and page, or give a Handbook reference. Remember that when you use a theorem, you must demonstrate that all the conditions of the theorem are satisfied.

The number of marks assigned to each part of a question is given in the right-hand margin, to give you a rough indication of the amount of time that you should spend on each part.

Your work should be written in a good mathematical style, as demonstrated by the exercise and worked exercise solutions in the study texts. You should explain your solutions carefully, using appropriate notation and terminology, defining any symbols that you introduce, and writing in proper sentences. There are no marks for how well you do this in this assignment, but there are five marks (referred to as 'good mathematical communication', or 'GMC', marks) for it in each subsequent TMA. Your tutor's comments on your solutions to this TMA (and later TMAs) will help you to gain the GMC marks in the later TMAs.

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You should read the information on the front page of this booklet before you start working on the questions.

Question 1 (Unit A1) – 13 marks

Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\},$$
$$B = \{(x,y) \in \mathbb{R}^2 : y \geq x + 1\},$$
$$C = \{(x,y) \in \mathbb{R}^2 : x + y \leq 4\}.$$

(a) Sketch, on separate diagrams, the sets A, B, $A \cup B$ and $B - A$. [5]

(b) Prove that A is a proper subset of C. [8]

Question 2 (Unit A1) – 12 marks

Let f and g be the functions defined by

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (y, -x)$$

and

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x+1, y-2).$$

(a) Describe the geometric effect of each of f and g. [2]

(b) Determine the composite function $g \circ f$. [2]

(c) Show that g is one-to-one and onto, and determine its inverse function g^{-1}. [8]

Question 3 (Unit A1) – 5 marks

Let P and Q be the points in $\mathbb{R}^3$ with position vectors $\mathbf{p} = (1,0,4)$ and $\mathbf{q} = (3,-1,5)$, respectively. Let O be the origin $(0,0,0)$.

(a) Find the vector form of the equation of the line l through P and Q. [2]

(b) Show that the point A with position vector $\mathbf{a} = (-1,1,3)$ lies on the line l, and that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{PQ}$. [3]

Question 4 (Unit A2) – 15 marks

(a) (i) Find the Cartesian form of the complex number $z = \dfrac{1+2i}{1-3i}$.

(ii) Determine the modulus and argument of z, and hence write down z in polar form. [5]

(b) Solve the following equations, expressing your answers in Cartesian form.

(i) $z^2 = i$

(ii) $z^3 - 4z^2 + 6z - 4 = 0$

Hint: In part (b)(ii), one of the solutions is a positive integer. [10]

Question 5 (Unit A3) – 20 marks

(a) (i) Write down the contrapositive of the following statement about integers:

If n^3 is a multiple of 3, then n is a multiple of 3.

(ii) Use the contrapositive to prove that the original statement is true. [6]

(b) Write down the converse of the following statement about real numbers:

$$x < 2 \implies x^2 < 4.$$

Determine whether the original statement is true, and whether the converse is true, justifying your answers. [6]

(c) Use mathematical induction to prove that

$$2 \times 1^2 + 3 \times 2^2 + \cdots + (n+1) \times n^2 = \frac{1}{12} n(n+1)(n+2)(3n+1)$$

for $n = 1, 2, \ldots$. [8]

Question 6 (Unit A3) – 10 marks

Consider the following statement.

Statement

The relation defined on $\mathbb{Z}$ by

$$m \sim n \quad \text{if } m^2 - n^2 \text{ is divisible by } 3$$

is an equivalence relation.

(a) Explain why the argument below is not a correct proof of the statement, identifying at least three errors. (There may be more than three errors, but you are required to identify only three. Your three errors should not include incorrect statements or omissions that follow entirely sensibly from earlier errors.) [4]

Proof (incorrect!)

$1 \sim 1$ since $1^2 - 1^2 = 0$, so the relation is reflexive.

$m^2 - n^2 = n^2 - m^2$, so if $m^2 - n^2$ is divisible by 3, then so is $n^2 - m^2$, and hence the relation is symmetric.

If $m \sim n$ and $n \sim p$, then $m^2 - n^2 = 3k$ and $n^2 - p^2 = 3k$, where $k \in \mathbb{Z}$. Hence $m^2 - p^2 = m^2 - n^2 + n^2 - p^2 = 6k$, so $m^2 - p^2$ is divisible by 3 and the relation is transitive.

Hence $\sim$ is an equivalence relation.

(b) Write out a correct proof of the statement. [6]

Question 7 (Unit A4) – 15 marks

Follow the graph-sketching strategy in Unit A4 to sketch the graph of the function

$$f(x) = \frac{6x}{x^2 + 4},$$

giving your working for each part of the strategy. *[15]*

Question 8 (Unit A4) – 10 marks

(a) Use scalings and/or translations of any of the basic graphs in Section 1 of Unit A4 to sketch the graphs of the following functions, explaining briefly how you obtained your graphs.

(i) $f_1(x) = |x + 2| - 3$

(ii) $f_2(x) = 2e^{-x} + 1$ *[7]*

(b) Hence sketch the graph of the function

$$f(x) = \begin{cases} |x + 2| - 3, & x \leq 0, \\ 2e^{-x} + 1, & x > 0. \end{cases}$$ *[3]*